OSCILLATIONS OF FIRST ORDER IMPULSIVE DELAY DIFFERENTIAL EQUATIONS

by

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First order impulsive delay differential equations are studied, where the fixed moments of impulsive effect (the jump points) are considered as up-jump points. Sufficient integral conditions for all solutions of these type of equations to be oscillatory are established.

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1. Introduction

Impulsive delay differential equations can model various processes and phenomena which depend on their prehistory and are subject to short-time disturbances. Such processes occur in the theory of optimal control, theoretical physics, population dynamics, pharmacokinetics, biotechnologies, industrial robotics, economics, etc. Starting from the work of Mil'man and Myshkis [9], in resent years there has been much current interest in studying of impulsive differential equations. Among numerous publications, we choose to refer to [1]-[12].

Consider the first order impulsive delay differential equations of the form

$$x'(t) + q(t)x(t) + p(t)x(t-h) = 0, \ t \neq \tau_k \qquad (E_1)$$

with the impulsive condition

$$\Delta x(\tau_k) = x(\tau_k + 0) - x(\tau_k - 0) = I_k(x(\tau_k), x(\tau_k - h)), \ k \in N$$

and with the initial condition

$$x(t) = \varphi(t), \ -h \leq t \leq 0; \ \varphi \in C([-h, 0]; R).$$

Here the delay $h > 0$ is a constant and $\tau_k \in (0, +\infty)$, $k \in N$ are fixed moments of impulsive effect (the jump points), which we characterize as *down-jumps* when $\Delta x(\tau_k) < 0$, $k \in N$ and as *up-jumps* when $\Delta x(\tau_k) > 0$, $k \in N$).

Denote by $PC(R,R)$ the set of all piecewise continuous on the intervals $(\tau_k, \tau_{k+1}], k \in N$ functions $u: R \to R$ which at the jump points $\tau_k, k \in N$ are continuous from the left, i.e. $u(\tau_k - 0) = \lim_{t \to \tau_k - 0} u(t) = u(\tau_k)$, and may have discontinuities of first kind at the jump points $\tau_k, k \in N$.

We also denote by $i[\tau_0, t)$ the number of fixed jump points $\tau_k \in [\tau_0, t),\ k \in N$, for $t > \tau_0$.

We clarify that

$$i[\tau_0, t) = \begin{cases} 0, \text{ for } \ \tau \in [\tau_0, \tau_1), \\ 0, \text{ for } \ \tau \in [\tau_1, \tau_2), \\ \ldots.. \\ k, \text{ for } \ \tau \in [\tau_k, \tau_{k+1}), \ k \in N. \end{cases}$$

Our aim is to establish sufficient conditions under which the equation $(E_1)$is oscillatory. In order to obtain our results, we need the following

Lemma 1 *Let $\tau_k,\ k \in N$ be fixed moments of impulsive effect (the jump points) with the property*

$$0 < \tau_1 < \tau_2 < \ldots < \tau_k < \ldots, \quad \lim_{t \to +\infty} \tau_k = +\infty$$

Then for every fixed $h > 0$ and for every $t \in [h, +\infty)$

$$M = \max_{t \in [h, +\infty]} i[t-h, t) < +\infty,$$

i.e. the number of the fixed moments of impulse effect $\tau_k \in [t-h, t),\ k \in N$ is finite.

Proof. Since, by the properties of the sequence $\tau_k,\ k \in N$, it follows that $\limsup_{k \to +\infty} \tau_k = +\infty$, we conclude that the only accumulation point of this sequence is that $+\infty$. Accordingly, for any number $T > 0$ there is an $n_0 \in N$ such that for every $n \geq n_0$ we have $\tau_n > T$. That is, the number of the fixed jump points in every finite interval of the form $[T-h, T)$ is a finite number. The proof of the lemma is complete.

Throughout this paper, unless otherwise mentioned, we will assume that the following hypotheses are satisfied:

(H_1) $0 < \tau_1 < \tau_2 < \ldots < \tau_k < \ldots, \lim_{k \to \infty} \tau_k = +\infty$ and

$0 < \min\{\tau_{k+1} - \tau_k\} \leq \max\{\tau_{k+1} - \tau_k\} < +\infty, k \in N$;

(H_2) The function $p : PC([0,\infty), (0,\infty))$ (resp. the function $q : PC([-h,\infty), R)$ with points of discontinuity $\tau_k, k \in N$, where it is continuous from the left, i.e. $p(\tau_k - 0) = p(\tau_k), k \in N$ (resp. $q(\tau_k - 0) = q(\tau_k), k \in N$);

(H_3) The function $I_k \in C(R^2; R)$ for all $v \in R$ and $k \in N$ has the following sign property

$$uI_k(u, v) > 0 \ for \ u \neq 0.$$

Moreover, the following notions will be used throughout this paper.

A continuous real valued function u defined on an interval of the form $[a, +\infty)$ eventually has some property if there is a number $b \geq a$ such that u has this property on the interval $[b, +\infty)$.

A real valued function u piecewise continuous on the set $[-h, \infty)\backslash\{\tau_k\}_{k=1}^{\infty}$ and continuous from the left at the jump points $\tau_k, k \in N$ with initial function $\varphi \in C([-h, 0]; R)$ is said to be a solution to $Eq.(E_1)$ if $u(t) = \varphi(t)$ for every $t \in [-h, 0]$ and $u(t)$ satisfies $Eq.(E_1)$ for all sufficiently large $t \geq 0$.

Without other mention, we will assume throughout that every solution $u(t)$ of $Eq.(E_1)$, that is under consideration here, is continuable to the right and is nontrivial. That is, $u(t)$ is defined on some ray of the form $[T_u, +\infty)$ and

$$\sup\{|u(t)|: \ t \geq T\} > 0 \text{ for each } T \geq T_u.$$

Such a solution is called a *regular solution* of $Eq.(E_1)$.

As usual, a regular solution of $Eq.(E_1)$ is called *nonoscillatory* if it is *eventually of constant sign* , i.e. if it is *eventually positive* or *eventually negative*. Otherwise, it is called *oscillatory*. Furthermore, $Eq.(E_1)$ is called *oscillatory* if every its regular solution is oscillatory. Otherwise, it is called *non-oscillatory*.

2. Main results

In order to achieve our goal, we begin our investigation with a special case of $Eq.(E_1)$. Namely, we consider the first order impulsive delay differential equation

$$x'(t) + p(t)x(t-h) = 0, \ t \neq \tau_k \tag{E_2}$$

with the impulsive condition

$$\Delta x(\tau_k) = x(\tau_k + 0) - x(\tau_k - 0) = I_k(x(\tau_k), x(\tau_k - h)), \ k \in N$$

and with the initial condition

$$x(t) = \varphi(t), -h \leq t \leq 0; \varphi \in C([-h, 0]; R),$$

which results from $Eq.(E_1)$ in the case where the function q is identically zero on the interval $[-h, \infty)$.

We start with the following

Lemma 2 *Let $x(t)$ be a non-oscillatory solution of $Eq.(E_2)$ and assume that the hypotheses (H_1) - (H_3) are satisfied. Suppose also that:*

(C_1) There is a positive constant L such that $\quad | I_k(u,v) | \leq L | u |$ for $u \neq 0, v \in R,\ k \in N$

and

$$(C_2)\ \liminf_{t\to\infty} \int_{t-h}^{t} p(s)ds \geq \frac{1}{e}(1+L)^{M},\ M = \max_{t\in[h,+\infty]} i[t-h,t).$$

Then $w(t)=\frac{x(t-h)}{x(t)}$ is an eventually bounded function.

Proof. Since the negative of a solution of $Eq.(E_2)$ is again a solution of $Eq.(E_2)$, it suffices to prove the lemma in the case of an eventually positive solution. So, suppose that $x(t)$ is an eventually positive solution of $Eq.(E_2)$. That is, there is a $t_0 \geq 0$ such that $x(t) > 0$ for $t \geq t_0$, while $x(t-h) > 0$ for $t \geq t_0 + h = t_1$. Therefore, from the impulsive condition of $Eq.(E_2)$, in view of the hypotheses (H_2) and (H_3), it follows that $x'(t) < 0$ and $\Delta x(\tau_k) > 0$ for $t, \tau_k \geq t_1, k \in N$ .Thus, $x(t)$ is a decreasing function on every interval $(\tau_k, \tau_{k+1}], \tau_k \geq t_1$, $k \in N$ and it has discontinuities of the first kind at the points of impulse effect $\tau_k \in R_+$, $k \in N$, considered as up-jumps.

Remark that, from the impulsive condition of $Eq.(E_2)$, using (C_1), we find

$$\frac{x(\tau_k+0)}{x(\tau_k)} = 1 + \frac{I_k(x(\tau_k), x(\tau_k - h))}{x(\tau_k)} \leq 1 + \frac{Lx(\tau_k)}{x(\tau_k)} = 1 + L,\ k \in N. \tag{1}$$

In order to prove our lemma, consider now the interval of integration $(t, t+\frac{h}{2})$, $t \geq t_1$ of $Eq.(E_2)$ and the number of the discontinuity points $i[t, t+\frac{h}{2})$ in it. Depending on the location of the points $t-h$ and t with respect to the jump points τ_k, $k \in N$, we distinguish the following five possible cases.

Case 1. When $t-h,\ t \in (\tau_k, \tau_{k+1}]$, $k \in N$ and exactly one of the following holds: either $i[t, t+\frac{h}{2}) = 0$ or else $i[t, t+\frac{h}{2}) = 1$.

Remark that, if $i[t, t+\frac{h}{2}) = 1$, then the only possible point of discontinuity in the interval $(t, t+\frac{h}{2})$ is the point τ_{k+1}.

In this case, integrating $Eq.(E_2)$ from t to $t+\frac{h}{2}$, $t \geq t_1 + \frac{h}{2}$, we obtain

$$x(t+\frac{h}{2}) - x(t) - \sum_{t\leq\tau_n\leq t+\frac{h}{2}} I_n(x(\tau_n), x(\tau_n - h)) + \int_{t}^{t+\frac{h}{2}} p(s)x(s-h)ds = 0,$$

and hence we find

$$x(t) + \sum_{t\leq\tau_n\leq t+\frac{h}{2}} I_n(x(\tau_n), x(\tau_n - h)) \geq \int_{t}^{t+\frac{h}{2}} p(s)x(s-h)ds \geq \int_{t_{M_1}}^{t+\frac{h}{2}} p(s)x(s-h)ds,$$

where $t_{M_1} = \max(t, max_{t\leq\tau_n\leq t+\frac{h}{2}}\tau_n)$. Observe that, if $i[t, t+\frac{h}{2}) = 0$, then $t_{M_1} = t$, while if $i[t, t+\frac{h}{2}) = 1$, then $t_{M_1} = \tau_{k+1}$. Now, applying the assumption *(C_1)* to the last inequality, we see that

$$x(t) + L \sum_{t\leq\tau_n\leq t+\frac{h}{2}} x(\tau_n) \geq x(t-\frac{h}{2}) \int_{t_{M_1}}^{t+\frac{h}{2}} p(s)ds$$

which implies that

$$x(t) + Lx(t)i[t, t+\frac{h}{2}) \geq x(t) + L \sum_{t\leq\tau_n\leq t+\frac{h}{2}} x(\tau_n) \geq x(t-\frac{h}{2}) \int_{t_{M_1}}^{t+\frac{h}{2}} p(s)ds,$$

and hence we get

$$\frac{x(t-\frac{h}{2})}{x(t)} \leq \frac{1 + Li[t, t+\frac{h}{2})}{\int_{t_{M_1}}^{t+\frac{h}{2}} p(s)ds}. \tag{2}$$

Next, integrating $Eq.(E_2)$ from $t - \frac{h}{2}$ to t, $t - \frac{h}{2} \geq t_1$, we see that

$$x(t) - x(t-\frac{h}{2}) + \int_{t-\frac{h}{2}}^{t} p(s)x(s-h)ds = 0$$

from where we obtain

$$x(t-\frac{h}{2}) \geq x(s-h) \int_{t-\frac{h}{2}}^{t} p(s)\ ds$$

and so we see that

$$\frac{x(t-h)}{x(t-\frac{h}{2})} \leq \frac{1}{\int_{t-\frac{h}{2}}^{t} p(s)ds} \tag{3}$$

In view of (2) and (3) and using the decreasing character of the function $x(t)$ on every interval $(\tau_k, \tau_{k+1}], \tau_k \geq t_1,\ k \in N$, we easily conclude that

$$1 < \frac{x(t-h)}{x(t)} \leq \frac{1 + Li[t, t+\frac{h}{2})}{\int_{t-\frac{h}{2}}^{t} p(s)ds \int_{t_{M_1}}^{t+\frac{h}{2}} p(s)ds}. \tag{4}$$

This shows that the function $w(t)$, $t \geq t_1$ is bounded and proves our assertion in *Case 1.*

Case 2. When $t-h, t \in (\tau_k, \tau_{k+1}],\ k \in N$ and $i[t, t+\frac{h}{2}) > 1$.

In this case, it is always possible to choose a sequence of points $\xi_l \in (\tau_k, \tau_{k+1}],\ l = 1, 2, ..., r$ with $\xi_1 = t - h$ and $\xi_r = t$, where for $h_{\xi_l} = \xi_l - \xi_{l-1}, l = 2, ..., r$, as in *Case 1*, exactly one of the following holds: either $i[\xi_l, \xi_l + \frac{1}{2}h_{\xi_l}) = 0$ or else $i[\xi_l, \xi_l + \frac{1}{2}h_{\xi_l}) = 1$. Then, for each pair ξ_{l-1}, ξ_l, $l = 2, ..., r$, as in the proof of *Case 1,* we obtain

$$1 < \frac{x(\xi_{l-1})}{x(\xi_l)} \leq \frac{1 + Li[\xi_l, \xi_l + \frac{1}{2}h_{\xi_l})}{\int\limits_{\xi_l - \frac{1}{2}h_{\xi_l}}^{\xi_l} p(s)ds \int\limits_{t_{M_{2,l}}}^{\xi_l + \frac{1}{2}h_{\xi_l}} p(s)ds}, \tag{5}$$

where $t_{M_{2,l}} = \max(\xi_l, max_{\xi_l \leq \tau_n \leq \xi_l + \frac{1}{2}h_{\xi_l}} \tau_n)$. In view of (5), we easily conclude that

$$1 \leq \frac{x(\xi_1)}{x(\xi_2)} \frac{x(\xi_2)}{x(\xi_3)} \cdots \frac{x(\xi_{r-1})}{x(\xi_r)} = \frac{x(t-h)}{x(t)} \leq \prod_{1 \leq l \leq r} \frac{1 + Li[\xi_l, \xi_l + \frac{1}{2}h_{\xi_l})}{\int\limits_{\xi_l - \frac{1}{2}h_{\xi_l}}^{\xi_l} p(s)ds \int\limits_{t_{M_{2,l}}}^{\xi_l + \frac{1}{2}h_{\xi_l}} p(s)ds}$$

which proves our assertion in *Case 2.*

Case 3. When $t \in (\tau_{k+1}, \tau_{k+2}]$, $t - h \in (\tau_k, \tau_{k+1}]$, $k \in N$ *and exactly one of the following holds: either* $i[t, t + \frac{h}{2}) = 0$ *or else* $i[t, t + \frac{h}{2}) = 1$.

Remark that, if $i[t, t + \frac{h}{2}) = 1$, then the only possible point of discontinuity in the interval $(t, t + \frac{h}{2})$ is the point τ_{k+2}.

In this case, because of the up-jump at the point τ_{k+1} ($\Delta x(\tau_{k+1}) > 0$ for $\tau_{k+1} \geq t_1$), depending on the value of $h > 0$ it is possible to have either (a) $x(t-h) \leq x(t)$ or (b) $x(t-h) \geq x(t)$.

If (a) holds, then (1) implies that

$$\frac{1}{1+L} \leq \frac{x(\tau_{k+1})}{x(\tau_{k+1}+0)} \leq \frac{x(t-h)}{x(t)} \leq 1 \tag{6}$$

which proves our claim in this case.

Assume now that (b) holds. In this case integrating $Eq.(E_2)$ from t to $t + \frac{h}{2}$, $t \geq t_1 + \frac{h}{2}$, and then from $t - \frac{h}{2}$ to t, $t - \frac{h}{2} \geq t_1$, as in the proof of *Case 1,* we derive (2) and

$$\frac{x(t-h)}{x(t-\frac{h}{2})} \leq \frac{1 + Li[t - \frac{h}{2}, t)}{\int\limits_{t_{M_3}}^{t} p(s)ds} \tag{7}$$

respectively, where $t_{M_3} = \max(t - \frac{h}{2}, max_{t - \frac{h}{2} \leq \tau_n \leq t} \tau_n)$. Remark that, if $i[t - \frac{h}{2}, t) = 0$, then $t_{M_3} = t - \frac{h}{2}$, while if $i[t - \frac{h}{2}, t) = 1$, then $t_{M_3} = \tau_{k+1}$.

By (2) and (7), taking into account the fact that $x(t-h) \geq x(t)$, we conclude that

$$1 \leq \frac{x(t-h)}{x(t)} \leq \frac{(1 + Li[t - \frac{h}{2}, t))(1 + Li[t, t + \frac{h}{2}))}{\int\limits_{t_{M_3}}^{t} p(s)ds \int\limits_{t_{M_1}}^{t+\frac{h}{2}} p(s)ds},$$

which is similar to (4) and proves our assertion in *Case 3.*

Case 4. When $t \in (\tau_{k+1}, \tau_{k+2}]$, $t-h \in (\tau_k, \tau_{k+1}]$, $k \in N$ *and* $i[t, t+\frac{h}{2}) > 1$.

Here, as in *Case 3,* it is possible to have either (a) $x(t-h) \le x(t)$ or (b) $x(t-h) \ge x(t)$. If (a) holds, then we derive (6) which proves our assertion. So, assume that (b) holds. In this case, it is always possible to choose a sequence $\eta_i \in (\tau_k, \tau_{k+1}]$, $i = 1, 2, ..., s$ with $\eta_0 = t-h$, and such that for $h_{\eta_i} = \eta_i - \eta_{i-1}$, $i = 1, 2, ..., s$, as in *Case 1*, exactly one of the following to be hold: either $i[\eta_i, \eta_i + \frac{1}{2}h_{\eta_i}) = 0$ or else $i[\eta_i, \eta_i + \frac{1}{2}h_{\eta_i}) = 1$. Then, as in the proof of *Case 1,* for each pair η_{i-1} and η_i, $i = 1, 2, ..., s$ we obtain

$$1 < \frac{x(\eta_{i-1})}{x(\eta_i)} \le \frac{1 + Li[\eta_i, \eta_i + \frac{1}{2}h_{\eta_i})}{\int\limits_{\eta_i - \frac{1}{2}h_{\eta_i}}^{\eta_i} p(s)ds \int\limits_{t_{M_{4,i}}}^{\eta_i + \frac{1}{2}h_{\eta_i}} p(s)ds} \tag{8}$$

where $t_{M_{4,i}} = \max(\eta_i, max_{\eta_i \le \tau_n \le \eta_i + \frac{1}{2}h_{\eta_i}} \tau_n)$.

Since $x(\eta_s) \in (\tau_k, \tau_{k+1}]$, we may choose a point $\xi_1 < t$ such that $\xi_1 \in (\tau_{k+1}, \tau_{k+2}]$, and, as in *Case 3,* η_s and ξ_1 for $h_1 = \xi_1 - \eta_s$ to satisfy exactly one of the following: either $i[\xi_1, \xi_1 + \frac{1}{2}h_1) = 0$ or else $i[\xi_1, \xi_1 + \frac{1}{2}h_1) = 1$.Then, for the pair η_s and ξ_1 , as in the proof of *Case 3,* we obtain

$$1 < \frac{x(\eta_s)}{x(\xi_1)} \le \frac{(1 + Li[\xi_1 - \frac{1}{2}h_1, \xi_1))(1 + Li[\xi_1, \xi_1 + \frac{1}{2}h_1))}{\int\limits_{t_{M_{4a}}}^{t} p(s)ds \int\limits_{t_{M_{4b}}}^{\xi_1 + \frac{1}{2}h_1} p(s)ds} = L_{\xi_1}(t) \tag{9}$$

where $t_{M_{4a}} = \max(\xi_1 - \frac{h_1}{2}, max_{\xi_1 - \frac{1}{2}h_1 \le \tau_n \le \xi_1} \tau_n)$, $t_{M_{4b}} = \max(\xi_1, max_{\xi_1 \le \tau_n \le \xi_1 + \frac{1}{2}h_1} \tau_n)$.

Now, in view of (8) and (9), we conclude that

$$1 \le \frac{x(\eta_0)}{x(\eta_1)} \frac{x(\eta_1)}{x(\eta_2)} ... \frac{x(\eta_s)}{x(\xi_1)} = \frac{x(\eta_0)}{x(\xi_1)} = \frac{x(t-h)}{x(\xi_1)} \le L_{\xi_1}(t) \prod_{1 \le i \le s} \frac{1 + Li[\eta_i, \eta_i + \frac{1}{2}h_{\eta_i})}{\int\limits_{\eta_i - \frac{1}{2}h_{\eta_i}}^{\eta_i} p(s)ds \int\limits_{t_{M_{4,i}}}^{\eta_i + \frac{1}{2}h_{\eta_i}} p(s)ds},$$

i.e. the function $\frac{x(t-h)}{x(\xi_1)}$ is bounded.

Finally, since the points ξ_1 and t with $\xi_1 < t$ belong to the same interval $(\tau_{k+1}, \tau_{k+2}]$, applying *Cases 1* or *Cases 2,* we prove that the function $\frac{x(\xi_1)}{x(t)}$ is also bounded.

So, from the above observation, it follows that the function $\frac{x(t-h)}{x(\xi_1)} \frac{x(\xi_1)}{x(t)} = \frac{x(t-h)}{x(t)} = w(t)$ for $t \ge t_1$ is bounded.

Case 5. When $t \in (\tau_{k+1}, \tau_{k+2}]$, *while* $t-h \in (\tau_{k-m}, \tau_{(k-m)+1}]$, $k \in N$ *for some fixed* $m \in \{1, 2, ..., M\}$, *where* $M = \max\limits_{t \in [h, +\infty]} i[t-h, t)$.

In this case for some fixed $m \in \{1, 2, 3, ...M\}$ we see that

$$\tau_{k-m} < t-h < \tau_{(k-m)+1)} < \tau_{(k-m)+2)} < ...\tau_k < \tau_{k+1} < t < \tau_{k+2}, \ k \in N$$

Let $\xi_0 = t-h$. Let also $\xi_j \in (\tau_{(k-m)+j}, \tau_{(k-m)+j+1}]$, $j=1,2,...,m+1$ be a sequence of points with $\xi_{m+1}=t$, for which exactly one of the previous cases holds. Then, for each pair ξ_{j-1}, ξ_j, $j=1,2,...,m+1$, as in the proofs of the previous cases considered above, we derive that each of the functions $\frac{x(\xi_{j-1})}{x(\xi_j)}$, $j=1,2,...,m+1$ is bounded. Therefore, the function $\frac{x(\xi_0)}{x(\xi_1)}\frac{x(\xi_1)}{x(\xi_2)}\cdots\frac{x(\xi_{m-1})}{x(\xi_m)}\frac{x(\xi_m)}{x(\xi_{m+1})} = \frac{x(t-h)}{x(t)} = w(t)$ for $t \geq t_1$ is also bounded.

The proof of the lemma is complete.

Now we state our first theorem which ensure that all solutions of $Eq.(E_2)$ are oscillatory.

Theorem 1 *Assume that the hypotheses (H_1) – (H_3) are satisfied. Suppose also that:*

(C_1) There is a positive constant L such that $|I_k(u,v)| \leq L|u|$ *for* $u \neq 0, v \in R,\ k \in N$

and

(C_2) $\liminf_{t\to\infty} \int_{t-h}^{t} p(s)ds \geq \frac{1}{e}(1+L)^{M}$, $M = \max_{t\in[h,+\infty]} i[t-h,t)$.

Then the equation (E_2) is oscillatory.

Proof. As in the proof of Lemma 2, we consider an eventually positive solution $x(t)$ of $Eq.(E_2)$ and a $t_1 \geq t_0 + h > 0$ such that $x(t) > 0$ and $x(t-h) > 0$ for $t \geq t_1$. Then, again as in the proof of Lemma 2, from the impulsive condition of $Eq.(E_2)$, using (C_1), we find

$$\frac{x(\tau_k+0)}{x(\tau_k)} = 1 + \frac{I_k(x(\tau_k), x(\tau_k-h))}{x(\tau_k)} \leq 1 + \frac{Lx(\tau_k)}{x(\tau_k)} = 1+L,\ k \in N. \tag{1}$$

Next, divide $Eq.(E_2)$ by $x(t)$, $t \geq t_1$ and integrate from $t-h$ to t to derive

$$\ln\frac{x(t-h)}{x(t)} + \sum_{t-h\leq\tau_k<t} \ln\frac{x(\tau_k+0)}{x(\tau_k)} = \int_{t-h}^{t} p(s)\frac{x(s-h)}{x(s)}ds,$$

where, by Lemma 2, $w(t) = \frac{x(t-h)}{x(t)}$, $t \geq t_1$ is a bounded function. From the above expression, in view of (1), we find

$$\ln w(t)(1+L)^{M} \geq \ln[w(t)\prod_{t-h\leq\tau_k<t}(1+L)] > w_l\int_{t-h}^{t} p(s)ds \tag{10}$$

where

$$w_l = \liminf_{t\to\infty} w(t), \qquad t \geq t_1.$$

Clearly, (10) implies that

$$\liminf_{t\to\infty}\int_{t-h}^{t} p(s)ds < \frac{1}{e}(1+L)^{M},$$

which contradicts (C_2). The proof of the theorem is complete.

As an immediate consequence of Theorem 1, we have the following

Corollary 1 *Suppose that all assumptions of Theorem 1 are satisfied. Then the corresponding to the equation* (E_2):

(a) inequality

$$x'(t) + p(t)x(t-h) \leq 0, \ t \neq \tau_k \tag{$N_{2,\leq}$}$$

$$\Delta x(\tau_k) \leq I_k(x(\tau_k), x(\tau_k - h)), \ k \in N$$

has no eventually positive solutions;

(b) inequality

$$x'(t) + p(t)x(t-h) \geq 0, \ t \neq \tau_k \tag{$N_{2,\geq}$}$$

$$\Delta x(\tau_k) \geq I_k(x(\tau_k), x(\tau_k - h)), \ k \in N$$

has no eventually negative solutions.

The proof of Corollary 1 is similar to the proof of Theorem 1 and so it is omitted.

Our next result concerns the oscillatory character of $Eq.(E_1)$. More precisely, we establish the following

Theorem 2 *Assume that the hypotheses* (H_1) – (H_3) *and* (C_1) *are satisfied. Suppose also that*

$$(C_3) \ \liminf_{t\to\infty} \int_{t-h}^{t} p(s)\exp(\int_{s-h}^{s} q(u)du)ds > \frac{1}{e}(1+L)^{M}.$$

Then the equation (E_1) *is oscillatory.*

Proof. Since the negative of a solution of $Eq.(E_1)$ is again a solution of $Eq.(E_1)$, it suffices to prove the theorem in the case of an eventually positive solution. So, suppose that $x(t)$ is an eventually positive solution of $Eq.(E_1)$. That is, there is a $t_0 \geq 0$ such that $x(t) > 0$ for $t \geq t_0$, while $x(t-h) > 0$ for $t \geq t_0 + h = t_1$. Set

$$x(t) = z(t)exp(\int_{0}^{t} q(s)ds), \ t \geq t_1. \tag{11}$$

Substituting (11) into $Eq.(E_1)$, we obtain

$$z'(t) + p_1(t)z(t-h) = 0, \ t \neq \tau_k; \tag{12}$$

with the impulsive condition

$$\Delta z(\tau_k) = J_k(z(\tau_k), z(\tau_k - h)), \ k \in N$$

where

$$p_1(t) = p(t) \ exp(\int_{t-h}^{t} q(s)ds), \ t \geq t_1$$

and

$$J_k(z(\tau_k), z(\tau_k - h)) = I_k(z(\tau_k)exp(-\int_0^{\tau_k} q(s)ds) \, , z(\tau_k - h)exp[-(\int_0^{\tau_k - h} q(s)ds)exp(\int_0^{\tau_k} q(s)ds)]), k \in N.$$

Since $Eq.(12)$ is of the form of $Eq.(E_2)$ and the functions p_1 and J_k, $k \in N$ satisfy the assumptions of Theorem 1, the conclusion of Theorem 2 is obvious.

Theorem 2 furnish the following

Corollary 2 *Suppose that all assumptions of Theorem 2 are satisfied. Then the corresponding to the equation* (E_1)*:*

(a) inequality

$$x'(t) + q(t)x(t) + p(t)x(t-h) \leq 0, \ t \neq \tau_k \tag{$N_{1,\leq}$}$$

$$\Delta x(\tau_k) \leq I_k(x(\tau_k), x(\tau_k - h)), \ k \in N$$

has no eventually positive solutions;

(b) inequality

$$x'(t) + q(t)x(t) + p(t)x(t-h) \geq 0, \ t \neq \tau_k \tag{$N_{1,\geq}$}$$

$$\Delta x(\tau_k) \geq I_k(x(\tau_k), x(\tau_k - h)), \ k \in N$$

has no eventually negative solutions.

The proof of Corollary 2 is similar to that of Theorem 2 and therefore it is omitted.

3. Examples

In order to illustrate the obtained results, we offer the following two examples .

Example 1 *Consider the impulsive delay differential equation*

$$x'(t) + \frac{5}{4}x(t-1) = 0, \; t \neq \tau_k, \; k \in N,$$

$$\Delta x(\tau_k) = \frac{1}{2}x(\tau_k) + x(\tau_k - 1), \; k \in N,$$

where $h = 1$ *and* $\tau_{k+1} - \tau_k = 1$. *In this case we have* $M = \max_{t\in[h,+\infty]} i[t-h,t) = 1$ *and especially for the assumptions* (C_1) *and* (C_2) *it is fulfilled*

$$\liminf_{t\to\infty} \int_{t-h}^{t} p(s)ds = \frac{5}{4} \geq \frac{1}{e}(1+L)^{\;M} \approx 1.1.$$

when

$$|\frac{1}{2}x(\tau_k) + x(\tau_k - 1)| \; \leq L|x(\tau_k)|, \; k \in N \quad for \; L \leq 2.$$

So, the assumptions of Theorem 1 are satisfied. Therefore, by Theorem 1, all solutions of the above equation are oscillatory. For example, the function

$$x(t) = e^{-\lambda_* t}A^{i[\tau_0,t)} \; with \; the \; initial \; function \; \varphi(t) = e^{-\lambda_*.t}, t \in [\tau_0 - 1, \tau_0],$$

where $\lambda_* = -1.9834$ *and* $A = -0.087$ *is an oscillatory solution of this equation.*

Example 2 *Consider the impulsive retarded differential equation*

$$x'(t) + \frac{1}{4}x(t-1) = 0, \; t \neq \tau_k, \; k \in N,$$

$$\Delta x(\tau_k) = -\frac{2}{10}x(\tau_k) + \frac{1}{10}x(\tau_k - 1), \; k \in N,$$

with $h = 1$ *and* $\tau_{k+1} - \tau_k = 1$. *In this case we have* $M = \max_{t\in[h,+\infty]} i[t-h,t) = 1$ *and it is easy to check that the assumption* (C_2) *is not satisfied, i.e.*

$$\liminf_{t\to\infty} \int_{t-h}^{t} p(s)ds = \frac{1}{4} < \frac{1}{e}(1+L)^{\;M} \; \approx \; 0.405$$

when

$$|-\frac{2}{10}x(\tau_k) + \frac{1}{10}x(\tau_k - 1)| \; \leq L|x(\tau_k)|, \; k \in N \quad for \; L \leq 0.1.$$

Hence, the above equation is non-oscillatory. That means that among its solutions at least one is non-oscillatory. In fact, the function

$$x(t) = e^{-\lambda_* t}A^{i[\tau_0,t)} \; with \; the \; initial \; function \; \varphi(t) = e^{-\lambda_*.t}, t \in [\tau_0 - 1, \tau_0], \tau_0 > 0$$

where $\lambda_* = 0.385$ *and* $A = 0.954$, *is a non-oscillatory solution of this equation. Remark that the above equation admits also oscillatory solutions. Such a solution is the function* $x(t) = e^{-\lambda_* t}A^{i[\tau_0,t)}$, *where* $\lambda_* = -2.04$ *and* $A = -0.016$.

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